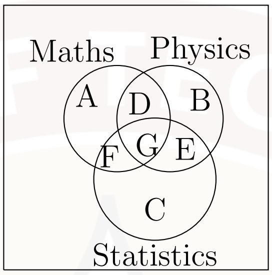
Week - 1
Solutions for Practice Assignment
Mathematics for Data Science - 1

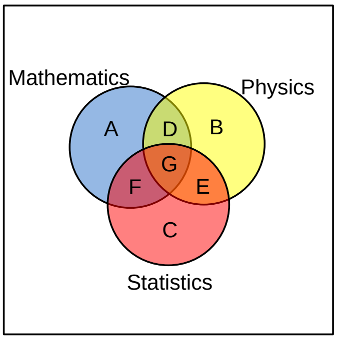
1. Given below is a Venn diagram for sets of students who take $Maths$, $Physics$, and $Statistics$. Which of the option(s) is(are) correct? [Notation: For sets P and Q, $P \setminus Q$ denotes the set of elements in P which are not in Q.]



- ◯ D is the set of students who take both $Maths$ and $Statistics$.
- ◯ $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects.
- ◯ E is a subset of the set of the students who have not taken $Maths$.
- ◯ $Maths \setminus D$ is the set of all students who have taken only $Maths$.
- ◯ $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$.

Solution: According to Figure 1, D is the set of students who take both $Maths$ and $Physics$. Hence the first statement is not valid.

The second option - $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects - is correct. This is because D is the set of students who take both $Maths$ and $Physics$, E is the set of students who take both $Physics$ and $Statistics$, F is the set of students who take both $Maths$ and $Statistics$ and G is the set of students who take all three subjects.



PS-1.1: Figure for Question 1

Third option - E is a subset of the set of the students who have not taken $Maths$ - is also correct. E is the set of students who take both $Physics$ and $Statistics$ and G is the set of students who take $Maths$ in addition to $Physics$ and $Statistics$. $(B \cup E \cup C)$ is the set of students who have not taken $Maths$. Clearly, E is a subset of this set. As E and G are two different sets, this option is correct.

Fourth option - $Maths \setminus D$ is the set of all students who have taken only $Maths$ - is not correct. $Maths \setminus D$ represents the students of $Maths$ who have not taken $Physics$ and may or may not have taken $Statistics$. This implies that students who take only $Maths$ (set A), or the students who take both $Maths$ and $Statistics$ (set F) or the students who take all three subjects (set G) are also included in $Maths \setminus D$ set. Hence this option is not correct.

Fifth option - $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$ - is correct. $(D \cup G \cup E)$ represents the students who take only $Maths$ and $Physics$ or all three subjects or $Physics$ and $Statistics$. $Physics \setminus (D \cup G \cup E)$ represents B, which is the set of students who only take $Physics$. Hence this option is correct.

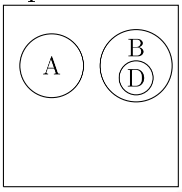
2. Let A be the set of natural numbers less than 6 and whose greatest common divisor with 6 is 1. Let B be the set of divisors of 6. What are the cardinalities of $A, B, A \cup B,$ and $A \cap B$?

- ◯ (1,5,6,0)
- ◯ (1,4,5,0)
- ◯ (2,4,5,1)
- ◯ (2,4,6,1)

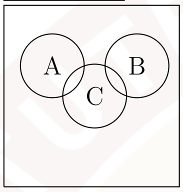
Solution: We have set A=$\{1, 5\}$, B=$\{1, 2, 3, 6\}$, $A \cup B$ =$\{1, 2, 3, 5, 6\}$ and $A \cap B$=$\{1\}$. It follows that the cardinalities (i.e. number of elements) of A, B, $A \cup B$ and $A \cap B$ are respectively 2, 4, 5 and 1. Hence, the third option - $\{2, 4, 5, 1\}$ - is correct.

3. Let A be the set of all even natural numbers (including zero), B be the set of all odd natural numbers, C be the set of all natural numbers which divide 100, and D be the set of all prime numbers less than 100. Which of the following is(are) correct representation of these sets? [Note: A region represented in a Venn diagram could be empty. Take the set of real numbers to be the universal set.]

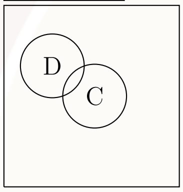
◯ Option 1



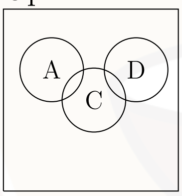
◯ Option 2



◯ Option 3



◯ Option 4



Solution: By definition, $A = \{0, 2, 4, 6, 8, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$, $C = \{2, 4, 5, 10, \ldots, 50\}$ and $D = \{2, 3, 5, 7, 11, .., 97\}$.
Option 1 shows D as a subset of all odd natural numbers. But D contains element 2, whereas B does not. Hence, this option is wrong.
Option 2 has overlap between A and C and overlap between B and C, but no overlap between A and B. A and B are sets of even and odd natural numbers which have no overlap. C is the set of natural numbers which divide 100. $A \cap C = \{2, 4, 10, 20, 50\}$ and $B \cap C = \{1, 5, 25\}$. Hence, this option is correct.
Option 3 represents C and D sets with an overlap between them. The overlapping area includes the set of all prime numbers which can divide 100. This is the set $\{2, 5\}$. Hence, option 3 is also correct.
$A \cap D = \{2\}$, but there is no overlap between A and D in Option 4. Hence, this option is wrong.

4. Let A be the set of natural numbers which are multiples of 5 strictly less than 100, and B be the set of natural numbers which divide 100. What are the cardinalities of the following sets?
$B \setminus A$ (the set of elements in B but not in A), $A \cap B$, and B

- ◯ (2, 5, 7)
- ◯ (4, 5, 9)
- ◯ (3, 4 , 7)
- ◯ (3, 5, 8)

Solution: By definition, $A = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$, $B = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$, $B \setminus A = \{1, 2, 4, 100\}$ and $A \cap B = \{5, 10, 20, 25, 50\}$. It follows that the cardinalities of sets $B \setminus A$, $A \cap B$ and B are, respectively, 4, 5 and 9. Hence, option 2 is correct.

5. Suppose the cardinality of set A is 2 and the cardinality of set B is 3, what are the cardinalities of the cartesian product $A \times B$ and the power set of $A \times B$?

- ◯ 6 and 36
- ◯ 5 and 32
- ◯ 6 and 64
- ◯ 5 and 25

Solution: Let the cardinality of set A be $\text{n}(A)$ and the cardinality of set B be $\text{n}(B)$. Then, the cardinality of the cartesian product $(A \times B)$, $\text{n}(A \times B) = \text{n}(A) \times \text{n}(B) = 3 \times 2 = 6$. If a set A has cardinality n, then the cardinality of power set of A is 2^n. It follows that the cardinality of the power set of $(A \times B)$ is $2^6 = 64$. Hence, the third option is correct.

6. In a survey, it is found that in a particular locality 64 houses buy English newspapers, 94 houses buy Tamil newspapers, and 26 houses buy both English and Tamil newspapers. How many houses buy newspapers of only one language?
   Answer: 106
   Solution: Number of houses which buy only English newspapers is (64 - 26) = 38.
   Number of houses which buy only Tamil newspapers is (94 - 26) = 68.
   Therefore, number of houses which buy either English or Tamil newspaper is
   (68 + 38) = 106.

7. Which of the following numbers is(are) irrational?

- ◯ $\underline{\sqrt{2+\sqrt{3}}}$
- ◯ $(2+\sqrt{3})(2-\sqrt{3})$
- ◯ $(2+\sqrt{3})+(2-\sqrt{3})$
- ◯ $\underline{2\sqrt{3}+3\sqrt{2}}$

Solution: Since $\sqrt{3}$ is an irrational number, it follows that $(2+\sqrt{3})$ and hence $\sqrt{(2+\sqrt{3})}$ are also irrational.
In the second option, $(2+\sqrt{3})(2-\sqrt{3}) = 4 - 3 = 1$, which is a rational number.
In the third option, $(2+\sqrt{3})+(2-\sqrt{3}) = 4$, which is also a rational number.
Since both $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers, we have $(2\sqrt{3}+3\sqrt{2})$ is an irrational number.

8. Which of the following is(are) true for the relation R given below?
$R = \{(a, b)|$ both a and b are even non-zero integers and $\frac{a}{b}$ is an integer $\}$

- ◯ R is a reflexive relation.
- ◯ R is a symmetric relation.
- ◯ R is a transitive relation.
- ◯ R is an equivalence relation.

Solution: A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$. R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and R is called transitive if (a, b) and (b, c) is in R implies $(a, c) \in R$. If a relation R is reflexive, symmetric and transitive, then it is called equivalence relation.
For any non-zero even integer a, $\frac{a}{a} = 1$ is an integer. Hence, $(a, a) \in R$, which implies that R is reflexive.
Now, let a = 4, and b = 2. Then, $\frac{a}{b} = \frac{4}{2} = 2$ is an integer. Hence, $(a, b) \in R$. But $\frac{b}{a} = \frac{2}{4} = \frac{1}{2}$ is not an integer. Therefore, $(b, a) \notin R$. It follows that R is not symmetric.
Let $(a, b) \in R$ and $(b, c) \in R$. That is, both $\frac{a}{b}$ and $\frac{b}{c}$ are integers. Hence, their product $\frac{a}{b} . \frac{b}{c} = \frac{a}{c}$ is also an integer. It follows that $(a, c) \in R$. Therefore, R is transitive.
Although R is reflexive and transitive but not symmetric, it is not an equivalence relation.

9. Find the domain and range of the following real valued function.
$f(x) = \sqrt{3-x}$ (Note: $\sqrt{}$ denotes the positive square root)

- ◯ domain=$\{x \in \mathbb{R} \mid x \neq 3\}$
  range=$\{x \in \mathbb{R} | x \geq 3\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \geq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \leq 0\}$

Solution: The set of real numbers $\mathbb{R}$ includes all rational and irrational numbers.
$\sqrt{a}$ is real valued if $a \geq 0$. If f has to be real valued, then

$$3 - x \geq 0$$

$$\Rightarrow 3 \geq x$$

Hence, domain of the function f is $\{x \in \mathbb{R} \mid x \leq 3\}$.
Since $\sqrt{}$ denotes the positive square root (as given in the question statement), the range of function f is nothing but all the positive real numbers, i.e. $\{x \in \mathbb{R} \mid x \geq 0\}$.

10. Which of the following is(are) true for the given function?
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 + 2$

- ◯ f is not injective.
- ◯ f is surjective.
- ◯ f is not surjective.
- ◯ f is bijective.

Solution: A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e. no two elements in the domain will have the same image. f is called surjective if for any element in the co-domain there is a pre-image in the domain, i.e. for any y in the co-domain, there exists an x in the domain such that $f(x) = y$. A function f is said to be bijective if it is both injective and surjective.
Since $f(x) = x^2 + 2$, we have $f(-1) = 3 = f(1)$. Hence, f is not injective. Now, the co-domain of the function is given as $\mathbb{R}$.
Now if f is surjective then codomain and the range should be same, that means every element in the codomain should have a preimage. Now let us try to find a preimage for 1 (observe that $1 \in \mathbb{R}$, as codomain of the function is given as $\mathbb{R}$). To find the preimage of 1, we have to find an element a from the domain for which $f(a) = 1$, i.e. $a^2 + 2 = 1$, i.e. $a^2 = -1$. Now we know that the square of any real number cannot be negative. Hence there cannot exist any real number a (in the domain) for which $f(a) = 1$. Hence 1 has no preimage. So codomain and range is not same. Hence f is not surjective. Also, $1 \in \mathbb{R}$. Let x be such that $x \in \mathbb{R}$, and $f(x) = 1$.
As the function is neither injective, nor surjective, therefore it is not bijective.

11. Find the domain of the following real valued function.
$f(x) = \frac{\sqrt{x+2}}{x^2-9}$

- ◯ $\underline{\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}}$
- ◯ $\{x \in \mathbb{R} \mid x \leq -2, x \geq 3\}$
- ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \leq 3\}$
- ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \neq 3\}$

Solution: $f(x) = \frac{\sqrt{x+2}}{x^2-9}$. For f to be a well-defined function, the denominator must be non-zero. That is,

$$x^2 - 9 \neq 0$$
$$\Rightarrow x \neq \pm 3$$

Further, if f has to be real valued, then $\sqrt{x+2}$ has to be real valued. Hence $x + 2$ must be non-negative. That is,

$$x + 2 \geq 0$$
$$\Rightarrow x \geq -2$$

It follows that the domain of the function $f(x)$ is $\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}$.

12. Let S be the set {January, February, March, April, May, June, July, August, September, October, November, December} of months in a year. Define the following three relations:

- $R_1 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same last four letters. } \}$
- $R_2 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same number of days. } \}$
- $R_3 := \{(a, c) \mid a, c \in S,\ \text{for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$

For example,(December, June)$\in R_3$ since (December, September)$\in R_1$, (September, June)$\in R_2$.

(a) Choose the correct option(s).

- ◯ R_3 is symmetric.
- ◯ R_3 is reflexive.
- ◯ R_3 is transitive.
- ◯ None of the above.

(b) What is the cardinality of R_3?
Answer: 85

Solution: For definitions of types of relations, please refer to solution of Question 8.
Every month has the same last four letters as itself (except *May* which has only three letters). In Table 1, the months whose name has been shown in red color have the same last four letters as each other. Similarly, the months whose name has been shown in blue color also have the same last four letters as each other.

Name of the months (Elements of S)
January
February
March
April
May
June
July
August
September
October
November
December

Table 1: Question 12 : R_1 relation

Hence R_1 = {(Jan, Jan), (Jan, Feb), (Feb, Jan), (Feb, Feb), (Mar, Mar), (April, April), (June, June), (July, July), (Aug, Aug), (Oct, Oct), (Sept, Sept), (Sept, Nov), (Sept, Dec), (Nov, Sept), (Nov, Nov), (Nov, Dec), (Dec, Sept), (Dec, Nov), (Dec, Dec)}

The relation R_2 consists of the pairs of months with the same number of days. In Table 2, the months whose name has been shown in red color have 31 days each. The months whose name has been shown in black color have 30 days each.

Name of the months
January
February
March
April
May
June
July
August
September
October
November
December

Table 2: Question 12: R_2 relation

Observe that it is a equivalence relation. The partition formed by this equivalence relation is as follows:

Class 1: Jan, Mar, May, July, Aug, Oct, Dec [Months with 31 days each]

Class 2: April, June, Sept, Nov [Months with 30 days each]

Class 3: Feb [Month with 28 or 29 days]

Now, R_3 is defined as follows:

$$R_3 = \{(a, c) \mid a, c \in S, \text{ for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$$

If $(a, c) \in R_3$, then there must exist some pair $(a, b) \in R_1$.
Let us list out the number of elements of R_3 by listing out pairs starting with as shown below :

January: (Jan, Jan) $\in R_1$, Now we assume three partitions in the set S, formed by the relation R_2. These partitions are class 1, class 2, class 3. Hence from these classes, 7 pairs will be there in R_3 starting with January. These are {(Jan, Jan), (Jan, Mar), (Jan, May), (Jan, July), (Jan, Aug), (Jan, Oct), (Jan, Dec)}. Moreover, (Jan, Feb) is in R_1, and Feb is in another partition in S due to R_2. So there are total 8 pairs (adding (Jan, Feb) with previous 7 elements) in R_3 starting with Jan.

February: Since (Feb, Jan) is in R_1, then due to class 1 there will be 7 pairs : {(Feb, Jan), (Feb, Mar), (Feb, May), (Feb, July), (Feb, Aug), (Feb, Oct), (Feb, Dec)}. The element (Feb, Feb) will be in R_3 due to class 3. Hence 8 pairs are there in R_3 starting with Feb.

March: Due to class 1, seven pairs {(Mar, Jan), (Mar, Mar), (Mar, May), (Mar, July), (Mar, Aug), (Mar, Oct), (Mar, Dec)}.

April: Due to class 2, four pairs {(April, April), (April, June), (April, Sept), (April, Nov)}.

May: No pair will start with May as there is no pair in R_1 starting with May.

June: Due to class 2, 4 pairs: {(June, April), (June, June), (June, Sept), (June, Nov)}

July: Due to class 1, 7 pairs. {(July, Jan), (July, March), (July, July), (July, Aug), (July, Oct), (July, Dec)}

August: Due to class 1, 7 pairs. {(Aug, Jan), (Aug, Mar), (Aug, May), (Aug, July), (Aug, Aug), (Aug, Oct), (Aug, Dec)}

September: As (Sept, Dec) is a pair in R_1, it will pair up will all months in class 1, and as (Sept, Sept) is in R_1, it will pair up with all months with class 2. Hence there are total 11 pairs in R_3 starting with Sept : {(Sept, Jan), (Sept, Mar), (Sept, May), (Sept, July), (Sept, Aug), (Sept, Oct), (Sept, Dec), (Sept, April), (Sept, June), (Sept, Sept), (Sept, Nov)}

October: Due to class 1, 7 pairs are there : {(Oct, Jan), (Oct, Mar), (Oct, May), (Oct, July), (Oct, Aug), (Oct, Oct), (Oct, Dec)}

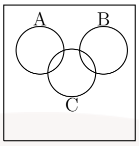
November: Due to both class 1 and class 2, 11 pairs : {(Nov, Jan), (Nov, Mar), (Nov, May), (Nov, July), (Nov, Aug), (Nov, Oct), (Nov, Dec), (Nov, April), (Nov, June), (Nov, Sept), (Nov, Nov)}

December: Due to both class 1 and class 2, 11 pairs: {(Dec, Jan), (Dec, Mar), (Dec, May), (Dec, July), (Dec, Aug), (Dec, Oct), (Dec, Dec), (Dec, April), (Dec, June), (Dec, Sept), (Dec, Nov)}

12. (b)
Hence cardinality of R_3 is $8 + 8 + 7 + 4 + 4 + 7 + 7 + 11 + 7 + 11 + 11 = 85$.
12. (a)

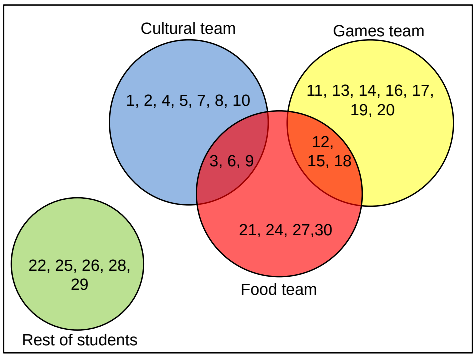
- (May, May) is not in R_3, hence R_3 is not reflexive.
- (Jan, May) is in R_3, but (May, Jan) is not in R_3, hence R_3 is not symmetric.
- (Mar, Dec) is in R_3, (Dec, Sept) is in R_3, but (Mar, Sept) is not in R_3. Hence R_3 is not transitive.

13. For a college event, thirty student volunteers were given id numbers from 1 to 30 such that each student gets a unique number. The students with id numbers from 1 to 10 are in Team 1 which coordinates the cultural program. The students with id numbers from 11 to 20 are in Team 2 which coordinates the games. The students whose roll numbers are multiples of 3 are in Team 3 which takes care of food. Now consider the following Venn diagram and choose the correct option(s).



- ◯ C, B , and A can represent Team 1, Team 2, and Team 3 respectively.
- ◯ A, B, and C can represent Team 1, Team 2, and Team 3 respectively.
- ◯ Roll number 15 has been assigned two jobs and is in both B and C.
- ◯ Roll number 25 is not in $A \cup B \cup C$.
- ◯ Roll number 10 is in both A and C.
- ◯ Cardinality of C is 20.

Solution:



PS-1.2: Venn diagram for Question 13

Figure PS-1.2 shows the Venn diagram corresponding to Question 13. Team 1, responsible for coordination of cultural programs, is represented by the blue circle. Team 2, responsible for game events, is represented by the yellow circle. Team 3, that takes care of food, is represented by the red circle. Rest of the students are represented using the green circle. Clearly, set A can correspond to the blue circle, B can denote the yellow circle and C can denote the red circle. That is, A, B, and C can represent Team 1, Team 2, and Team 3 respectively. Hence, option 2 is correct and option 1 is wrong. Roll number 15 is a common element between games team and food team, hence, option 3 is correct. Roll number 25 is located in the range of students with Roll number 21 to 30 but 25 is not divisible by 3. Hence, 25 does not belong to the set $A \cup B \cup C$ and so option 4 is correct. The number 10 is not divisible by 3, hence Roll number 10 is not in the set C. Therefore, option 5 is wrong. Further, since cardinality of C is 10, option 6 is also wrong.